Week 4

L^p-space : function version of ℓ^p

vector space of continuous functions on I

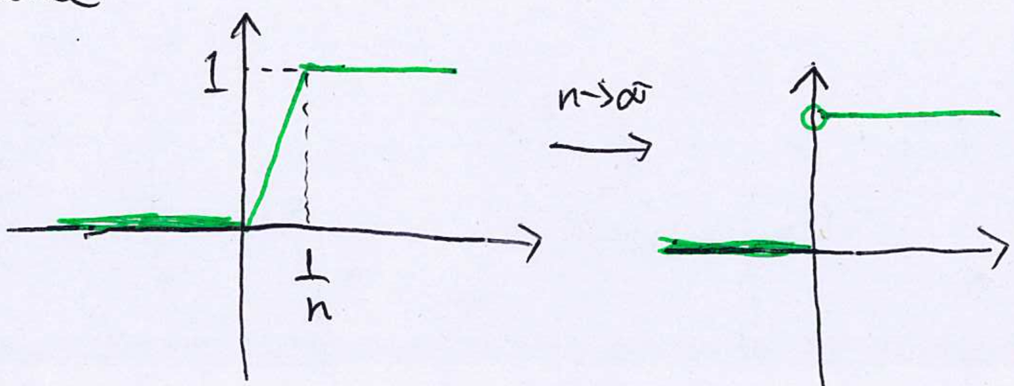
Let $f \in \mathcal{C}(I)$, where $I \subseteq \mathbb{R}$ is a bounded interval

Define $\|f\|_p = \left(\int |f(x)|^p \, dx \right)^{\frac{1}{p}}$

Problem: $\mathcal{C}(I)$ is not complete under this norm

eg. $f_n(x) = \begin{cases} nx & x \in [0, \frac{1}{n}] \\ 1 & x \in (\frac{1}{n}, 1] \\ 0 & x \in [-1, 0) \end{cases}$ $\quad I = [-1, 1]$

a Cauchy sequence

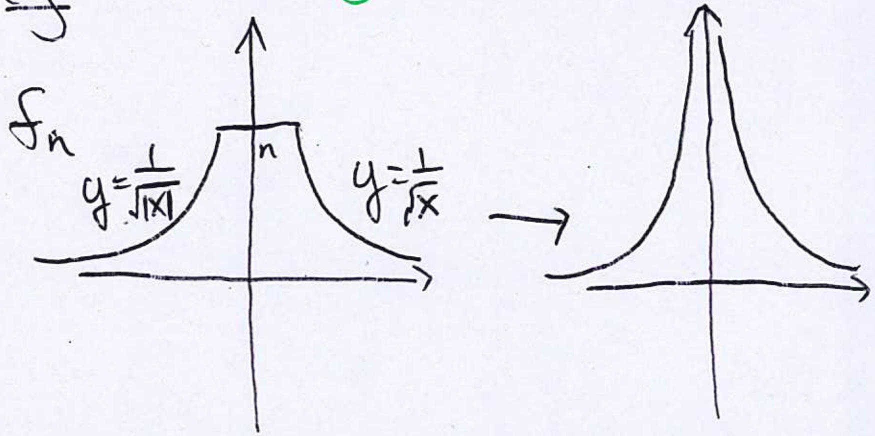


f_n is Cauchy sequence

but $f_n \not\to$ any continuous f in $\|\cdot\|_p$-norm

does not converge

eg

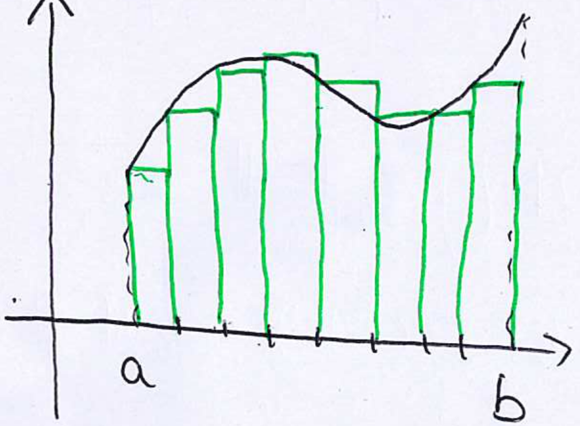


Continuous	Not continuous
	undefined at 0
Bounded	Unbounded

Need a good theory of integration for discontinuous functions and unbounded functions

Two definitions of integral

Riemann integral



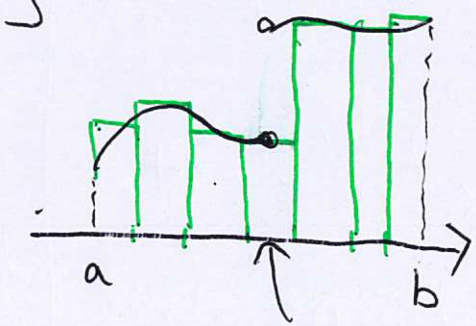
$$\sum_n f(x_i)\Delta x_i \xrightarrow{n\to\infty} \int_a^b f(x)dx$$

f(x) is said to be Riemann integrable

if the limit exists with $|\Delta x_i| \to 0$

- Easy
- Good for bounded function with "few" discontinuity

eg



not too tall

"very thin" after taking limit

Limitation of Riemann integral

- Not good for unbounded functions or "very discontinuous" functions

Riemann integrability

Measure zero set

Defn Let $A \subseteq \mathbb{R}$. A is said to be measure zero if $\forall \varepsilon > 0$, $\exists$ finite/countable many intervals (a_i, b_i) such that

① $A \subseteq \bigcup (a_i, b_i)$

② $\sum_i (b_i - a_i) < \varepsilon$

eg 1 Finite set in $\mathbb{R}$ is measure zero

Let $S = \{x_1, x_2, x_3, \cdots, x_k\} \subseteq \mathbb{R}$. For $\varepsilon > 0$,

let $J_i = \left(x_i - \frac{\varepsilon}{3k}, x_i + \frac{\varepsilon}{3k}\right) \leftarrow$ length $\frac{2\varepsilon}{3k}$

Then ① $S \subseteq \bigcup J_i$

② $\sum_i |J_i|^{\leftarrow \text{length}} = k\frac{2\varepsilon}{3k} = \frac{2}{3}\varepsilon < \varepsilon$

eg 2 Infinite countable set in $\mathbb{R}$ is measure zero

Let $S = \{x_1, x_2, x_3, x_4, \cdots\cdots\}$

For $\varepsilon > 0$. Let $J_i = \left(x_i - \frac{\varepsilon}{2^{i+2}}, x_i + \frac{\varepsilon}{2^{i+2}}\right)$

Then ① $S \subset \bigcup J_i$

② $\sum_i |J_i| = \sum_i \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2} < \varepsilon$

eg 3 Finite/countable union of measure zero set is measure zero

eg 4 $[0,1]$ is not measure zero

How to check? Use compactness

Def f is said to have property P almost everywhere if

$$\{x : f \text{ does not have property } P \text{ at } x\}$$

is measure zero.

Criterion for Riemann integrability

Let $f : [a,b] \to \mathbb{R}$.

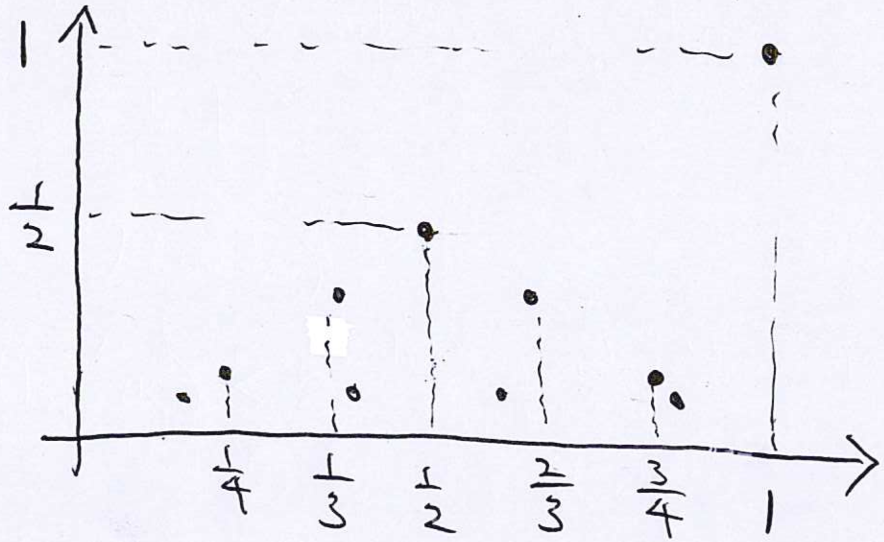
f is Riemann integrable

$$\Leftrightarrow \begin{cases} f(x) \text{ is bounded} \\ f(x) \text{ is } \underbrace{\text{continuous almost everywhere}}_{} \end{cases}$$

i.e. f is discontinuous only at a measure zero set

eg (Dirichlet function) $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ are coprime}, \ q > 0 \\ 0 & \text{otherwise} \end{cases}$$



$f(x) \neq 0$ if $x \in \mathbb{Q}$

$= 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$

Also, f is bounded on $[0,1]$

Fact f is continuous at $[0,1] \setminus \mathbb{Q}$

discontinuous at $[0,1] \cap \mathbb{Q}$ measure zero

$\Rightarrow f$ is Riemann integrable

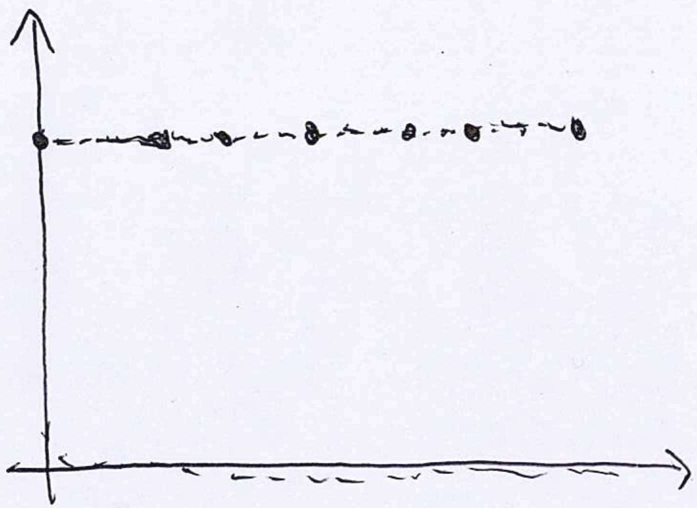
eg $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

f is not continuous anywhere

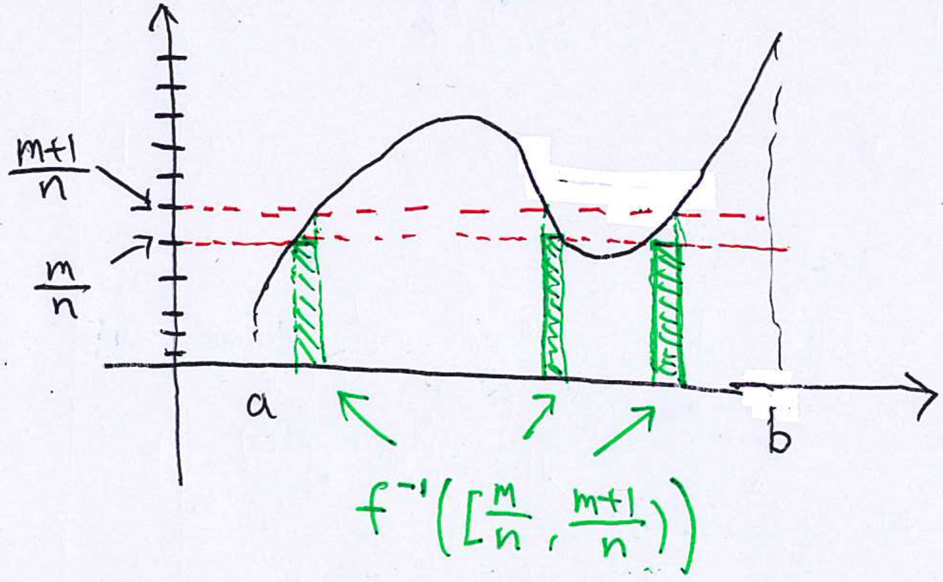
$[0,1]$ is not measure zero

$\Rightarrow$ f is not Riemann integrable on $[0,1]$



Lebesgue integration

Picture For a non-negative function f



$$\sum_{m=0}^{\infty} \frac{m}{n} \mu\left(f^{-1}\left(\left[\frac{m}{n}, \frac{m+1}{n}\right)\right)\right)$$

"measure" (size of the set)

$$\xrightarrow{\text{Limit } n \to \infty} \int f(x)\, dx$$

f is said to be Lebesgue integrable if limit exists

Fact

- If f is Riemann integrable, then f is Lebesgue integrable. Moreover, they have same integral value

- Some unbounded functions are Lebesgue integrable

eg $f(x) = \frac{1}{\sqrt{x}}$ on $(0,1)$

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \left[2\sqrt{x}\right]_0^1 = 2$$

f is not Riemann integrable (unbounded) but is Lebesgue integrable

Reference for Riemann / Lebesgue integral

R.G. Bartle & D.R. Sherbert

Introduction to Real Analysis ← Riem

W. Rudin

Principles of Mathematical Analysis

H.L. Royden

Real Analysis ← Lebesgue, measure theory

L^p- space

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ $(1 \le p < \infty)$

Let $I \subseteq \mathbb{R}$ be an interval

Let $L^p(I)$ (or just L^p if understood)

be the set of all "measurable" functions f

a very mild condition

such that

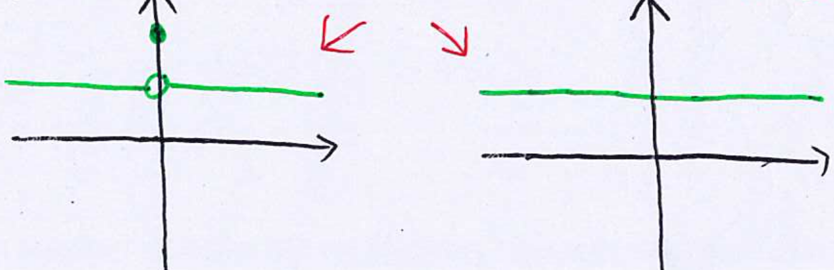
$$\int_I |f(x)|^p \, dx < \infty$$

where f and g are identified if

$f = g$ almost everywhere

eg.

Considered to be same functions

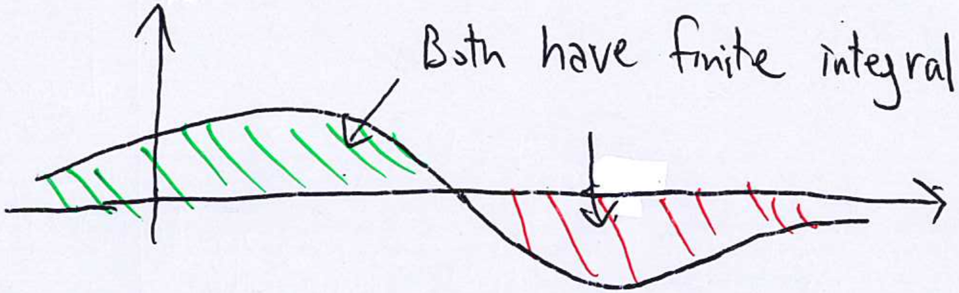


If $f \in L^p$, define

$$\|f\|_p = \left(\int_I |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

Rmk

① Absolute value ensures



② If $f = g$ a.e. (almost everywhere)

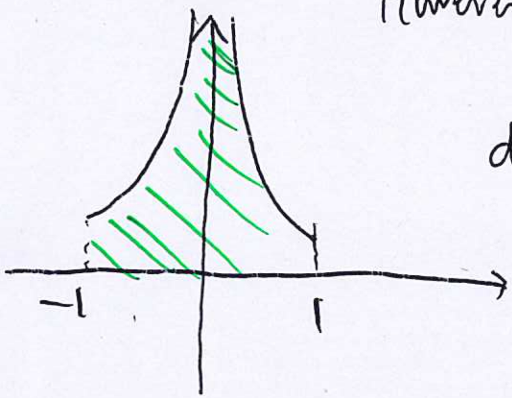
then $\int f(x)\,dx = \int g(x)\,dx$

③ f may not be defined everywhere on I

(almost everywhere is OK!)

e.g. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\,dx = 4 \leftarrow$ the integral makes sense

However, $\frac{1}{\sqrt{|x|}}$ is not defined at $x=0$



④ We may replace I by any "measurable" subset $A \subseteq \mathbb{R}$

For $p=\infty$,

Let $L^\infty(A)$ be the set of all "measurable" functions f such that

$\exists M>0$ with $|f(x)| \leq M$ a.e.

where f,g are identified if $f=g$ a.e.

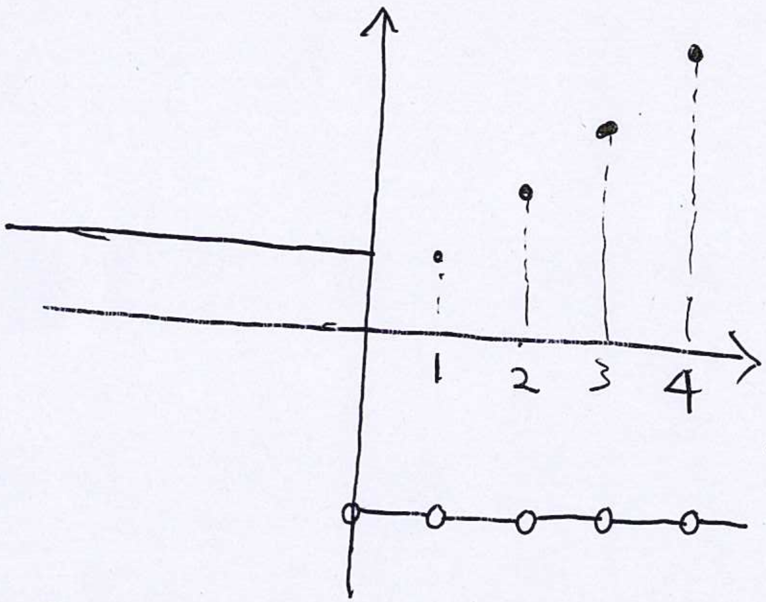
Define

$$\|f\|_\infty = \inf\{M>0 : |f(x)| \leq M \text{ a.e}\}$$

f is said to be essentially bounded if $f \in L^\infty(A)$

eg

$$f(x) = \begin{cases} x & \text{if } x = 1,2,3,4,\cdots \\ 1 & \text{if } x \leq 0 \\ -2 & \text{otherwise} \end{cases}$$



f is unbounded but essentially bounded

$f \in L^\infty$

$\|f\|_\infty = 2$

Note on L^p and L^∞-space

Hölder

Let $p, q > 1$ $\quad \frac{1}{p} + \frac{1}{q} = 1$

Let $f \in L^p$ $\quad g \in L^q$ then

$$\int |f(x) g(x)| \, dx \leq \left(\int |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int |g(x)|^q dx \right)^{\frac{1}{q}}$$

ie. $\quad fg \in L^1$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Minkowski

Let $p \geq 1$, $f, g \in L^p$

Then

$$\left(\int |f(x)+g(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\int |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int |g(x)|^p dx\right)^{\frac{1}{p}}$$

i.e. $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

Pf Similar to l^p, l^∞ versions

Replace infinite sum by integration

Consequence

$(L^p, \|\cdot\|_p)$ $\quad 1 \leq p \leq \infty$

is a complete normed space

i.e. Banach space

Defn A Banach space is a complete normed space

Fact For $a < b$

$$C[a,b] \overset{\text{dense}}{\subseteq} L^p[a,b]$$

$$\underbrace{\overline{C[a,b]}}_{\text{closure}} = L^p[a,b]$$

Bounded Linear Operators

Defn Let X, Y be normed spaces

A linear operator (ie linear transformation)

$$T : \underbrace{\mathcal{D}(T)}_{\text{Domain of } T} \longrightarrow Y \quad \text{where } \mathcal{D}(T) \subseteq X$$

is said to be bounded if $\exists\, c \geq 0$ s.t.

$$\underbrace{\|T(x)\|}_{\text{norm in } Y} \leq c \underbrace{\|x\|}_{\text{norm in } X} \quad \forall x \in \mathcal{D}(T)$$

Let $B(X,Y)$ be the set (vector space) of all bounded linear operators from X to Y.

Note

$$\|T(x)\| \leq c\|x\| \quad \forall x \in \mathcal{D}(T)$$

$$\iff \frac{\|T(x)\|}{\|x\|} \leq c \quad \forall x \in \mathcal{D}(T) \text{ with } x \neq 0$$

Defn For a bounded linear operator T

Define $$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|}$$

$\|T\|$ is called the norm of T

Rmk $$\boxed{\|T(x)\| \leq \|T\|\,\|x\|}$$

2.7-2 Lemma

Let T be a bounded linear operator. Then

(a)
$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\|=1}} \|T(x)\|$$

(b) $\|T\|$ defines a norm on $B(X,Y)$